

The two-level atom laser: analytical results and the laser transition.

Paul Gartner

*Institute for Theoretical Physics,
University of Bremen, 28334 Bremen, Germany
and*

*National Institute of Materials Physics,
Bucharest-Măgurele, Romania**

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The problem of the two-level atom laser is studied analytically. The steady-state solution is expressed as a continued fraction, and allows for accurate approximation by rational functions. Moreover, we show that the abrupt change observed in the pump dependence of the steady-state population is directly connected with the transition to the lasing regime. The condition for a sharp transition to Poissonian statistics is expressed as a scaling limit of vanishing cavity loss and light-matter coupling, $\kappa \rightarrow 0$, $g \rightarrow 0$, such that g^2/κ stays finite and $g^2/\kappa > 2\gamma$, where γ is the rate of atomic losses. The same scaling procedure is also shown to describe a similar change to Poisson distribution in the Scully-Lamb laser model too, suggesting that the low- κ , low- g asymptotics is of a more general significance for the laser transition.

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I. INTRODUCTION

A single two-level emitter in interaction with a cavity mode is the simplest illustration of cavity quantum electrodynamics. Actually, in experimental situations the emitter, either a true atom or an artificial one (quantum dot), usually brings into the picture more than just two levels. Besides the pair of states which couple with the laser mode, other states are involved in the carrier excitation and scattering processes. It was early realized [1] that incoherent pumping directly from the lasing levels destroys the coherence between these and the photon output is strongly reduced for large pump rates. This quenching effect acts against lasing and can be avoided by involving additional states. More recent realizations of single-emitter lasers [2, 3] were modelled using at least four levels [4, 5]. In a semiconductor quantum dot (QD) one typically encounters many-level, many-carrier situations which give rise to multiexcitonic effects. Moreover, in the presence of a spectral continuum (e.g. wetting-layer extended states) where, in typical experimental situations carriers are initially created by optical or electrical pumping, the creation of fermionic reservoirs becomes possible, from which carriers are injected into the QD [7]. The same charge distributions in the continuum are also a source of screening and additional dephasing [6]. All these effects are clearly beyond the reach of a two-level description.

Despite this, the two-level emitter model still enjoys a wide popularity due to its simplicity and transparency and remains a subject of intense investigations [8, 9]. Also, models with more levels are sometimes mapped into effective two-level ones [10, 11]. Many aspects of the light-matter interaction, like spectral properties as-

sociated with strong coupling, are quite well described in this framework [8, 9, 12–15].

An appealing feature of the two-level model is the fact that the carrier degrees of freedom are easily eliminated [16] leading to closed equations for the purely photonic statistics. This is a feature shared with the Scully-Lamb random injection model [17], which addresses directly the photonic density matrix. Therefore these cases are particularly appropriate for the study of the laser transition, and indeed, the Scully-Lamb model became a paradigm and a textbook case in this respect [18–21]. In comparison, with a few exceptions [22, 23], the two-level emitter has not enjoyed the same attention from this point of view.

In the Scully-Lamb model it is shown that the transition to lasing corresponds to the peak of the photon number distribution getting detached from the origin and moving away to larger and larger values. A thermal, exponential distribution changes into a Poissonian statistics. In terms of the Glauber-Sudarshan representation [24], this means that the so-called quasi-distribution function, initially located around the origin of the complex plane, becomes narrowly concentrated on a ring of increasing radius. Such a behaviour was also obtained numerically [1, 25] or by using analytic approximations [24], in several different contexts.

In the case of the two-level model, the numerical simulations show an abrupt change of regime, most clearly seen on the steady-state population behaviour: the strictly concave dependence of the upper-level population on the pumping rate becomes with a high accuracy linear, in the interval of intermediate pump rates [10, 15, 22]. Various numerical evidence suggest that this transition is associated with the onset of lasing, but the connection to the behaviour of the Glauber-Sudarshan function, like in the random injection model, is far from obvious.

* gartner@itp.uni-bremen.de

In the present paper we address the two-level laser problem by analytic methods. We show that the steady-state solution, usually obtained by the long-time limit of the Liouville-von Neumann evolution, can be expressed directly as a continued fraction involving the model parameters. Exact relations as well as simple, yet accurate approximations for the solution can be derived in some cases from the full formula. The theory of continued fractions also provides error control for these approximations.

But what is more important, the method also allows for a detailed characterization of the laser transition. The appearance of a jump in the solution (or some derivative) has its analog in the problem of phase transitions, to which lasing was sometimes compared, and the role of exactly soluble models is to clarify the mechanism of the jump.

The two-level atom turns out to be such an exactly soluble, yet nontrivial example on which the laser transition can be studied. The condition for the appearance of the above-mentioned abrupt change in the pump dependence of the population is proven to be equivalent to the vanishing of the Glauber-Sudarshan function at the origin. Moreover, one can show that this vanishing requirement is met only in a certain scaling limit for the model parameters, and then it also entails a Poissonian photon statistics.

Similarities with the thermodynamic limit in phase transitions and appropriate scaling asymptotics, like the Grad limit or the weak-coupling, long-time limit, in the theory of markovian kinetics [26] come to mind. In all these examples a regime change occurs, rigorously speaking, only in a certain asymptotic parameter domain, and this is precisely what takes place in our case too. Of course, with parameters close but slightly away from the scaling limit the transition may still be seen, albeit less sharply.

The paper is organized as follows: After defining the model (Sec. II) the three-term recursion equation obeyed by an infinite sequence of photon expectation values [16] is discussed and its solution expressed as a continued fraction. It is shown that *outside the interval of linear behaviour* the continued fraction converges rapidly and its truncation provides simple and accurate approximants (Sec. III). The discussion of the linear regime is more complicated and requires an extension of the recursion relation. This is achieved by noting that the photonic expectation values are moments of the Glauber-Sudarshan function, and it is shown that the linear behaviour is equivalent to the condition that this function vanishes at the origin (Sec. IV).

The next sections deal with the circumstances in which this condition is met. Heuristic suggestions in this sense are obtained using the rate-equation approximation (Sec. V). Turning back to the full problem (Sec. VI) it is shown that exact vanishing occurs as a limiting case, in which the rate of cavity losses κ and the light-matter coupling strength g go simultaneously to zero so that $g^2/\kappa = \text{const.}$ A similar scaling limit was obtained by

Rice and Carmichael, as a condition for a sharp lasing transition in the rate-equation approach to the many-emitter laser [27]. It should be noted that in contradistinction to this many-emitter result, in the present case, as the pump increases the system undergoes two transitions, a first one at the onset of lasing and a second one at larger pump values, when the lasing is destroyed by quenching. It is also shown that in this limiting procedure, in the lasing regime the Glauber-Sudarshan function not only vanishes at the origin but also becomes δ -like distributed on a ring centered at the origin, in agreement with the expected Poissonian photon statistics. The radius of this ring starts increasing from zero when the pump reaches the threshold value but eventually decreases back to zero due to quenching.

The conclusions regarding the scaling limit and the laser transition go beyond the particular case of the two-level laser, as checked on the Scully-Lamb model (see Appendix B).

II. THE MODEL AND ITS STEADY-STATE BEHAVIOUR

We consider a two-level emitter (atom or QD) in Jaynes-Cummings (JC) interaction with a cavity mode. Cavity losses, spontaneous recombination into non-lasing modes and incoherent pumping are included in the density operator evolution through Lindblad terms with rates κ , γ and P , respectively. Assuming perfect resonance between the emitter and the cavity mode one has, in the interaction representation, the following Liouville-von Neumann equation (in $\hbar = 1$ units)

$$\begin{aligned} \frac{\partial}{\partial t}\rho = & -i[H_{JC}, \rho] + \frac{\kappa}{2} [2b\rho b^\dagger - b^\dagger b\rho - \rho b^\dagger b] \\ & + \frac{\gamma}{2} [2v^\dagger c\rho c^\dagger v - c^\dagger c\rho - \rho c^\dagger c] \\ & + \frac{P}{2} [2c^\dagger v\rho v^\dagger c - v^\dagger v\rho - \rho v^\dagger v] \end{aligned} \quad (1)$$

The corresponding interaction Hamiltonian is given by

$$H_{JC} = g [b^\dagger v^\dagger c + b c^\dagger v] . \quad (2)$$

The creation and annihilation operators are b^\dagger, b for the laser mode, c^\dagger, c for the upper (conduction band) level and v^\dagger, v for the lower (valence band) one. Only one carrier is present in the system, so that one has $c^\dagger c + v^\dagger v = 1$. We use QD notations throughout, the atomic pseudospin formalism being easily recovered via $\sigma = v^\dagger c$, $\sigma^\dagger = c^\dagger v$, $\sigma^\dagger \sigma = c^\dagger c$, a.s.o.

The Liouville-von Neumann equation can be solved numerically until the steady-state solution is reached. Then various steady-state expectation values, regarding level populations and photon statistics can be obtained as a function of the pump rate P or other parameters [8–10].

Examining families of such plots [8] one clearly detects two types of behaviour. One example of each of

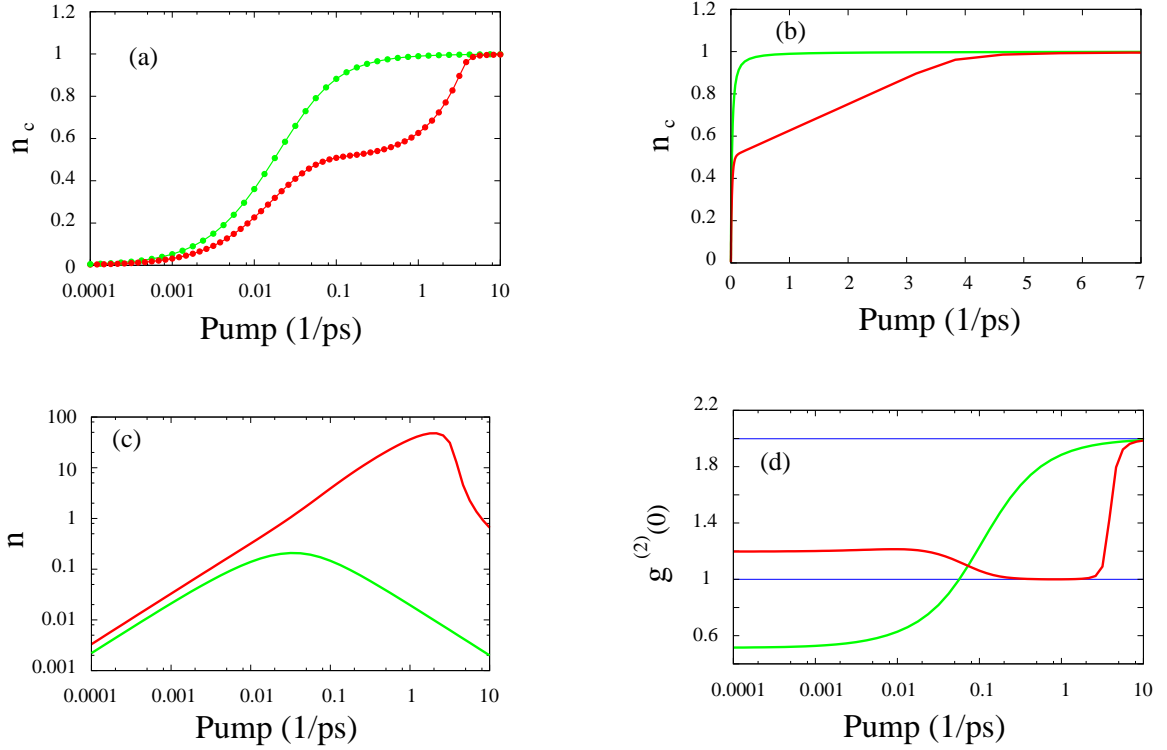


FIG. 1. Steady state solutions of the Liouville-von Neumann equation vs. pump rate P . The parameters are: Set A (red lines): $\kappa = 0.01/\text{ps}$, $g = 0.1/\text{ps}$, $\gamma = 0.02/\text{ps}$ and Set B (green lines): $\kappa = 0.02/\text{ps}$, $g = 0.01/\text{ps}$, $\gamma = 0.01/\text{ps}$. (a) the upper state population n_c , semi-log plot. Circles correspond to the solution obtained by continued fraction (see Sec III). (b) the same in linear plot, showing the appearance of the linear middle regime. (c) average photon number n . (d) second-order autocorrelation function.

these types is shown in Fig.1 (see the Caption for the parameters used). The two cases are best distinguished by their pump dependence of the upper-level population $n_c = \langle c^\dagger c \rangle$. In a semi-logarithmic plot one may have, depending on the parameters, either a simple S-shaped curve or a double S-shaped one (Fig.1a). The difference is more striking in a linear plot (Fig.1b), where in the first case one has a strictly concave function of P , while in the second there appears an almost perfectly linear shortcut in the middle P -region. This is strongly reminiscent of the behaviour of thermodynamic potentials in the presence of a phase transition [28].

In the present case too, the transition from concave to linear dependence in the middle P interval is a clear symptom of an abrupt regime change in the functioning of the two-level laser. The figure shows that this “linear middle” regime (as it will be subsequently called) is associated with inversion of population ($n_c > 1/2$), a large photon output $n = \langle b^\dagger b \rangle$ (Fig.1c), and a value of the second-order photon autocorrelation function at zero delay $g^{(2)}(0) = \langle b^\dagger b^\dagger b b \rangle / \langle b^\dagger b \rangle^2$ close to unity (Fig.1d). All these are indicative of the onset of lasing which, depending on the parameters, may take place or not, as illustrated by the two types of behaviour of Fig.1.

In what follows it will be proven that the linear pump

dependence of the population is indeed associated with a Poissonian photon statistics, characteristic for coherent light. At the same time, the conditions for the presence or absence of the transition, as well as cases in which the transition becomes sharp will be identified.

III. SOLUTION BY CONTINUED FRACTIONS

It is easy to see that the Liouville-von Neumann equation leads to a closed system of equations for a subset of density matrix elements. In the basis of QD-states $|c\rangle, |v\rangle$ and photon number n , these are the diagonal ones $\rho_{n,n}^{cc}, \rho_{n,n}^{vv}$ and only those off-diagonal elements between states having the same number of excitations $c^\dagger c + b^\dagger b$, i.e. $\rho_{n,n+1}^{cv}$ and their complex conjugates $\rho_{n+1,n}^{vc}$. The latter restriction is a consequence of the JC Hamiltonian, which conserves the excitation number.

Equivalently, one may limit oneself to the following set of expectation values

$$\begin{aligned} c_k &= \langle c^\dagger c b^{\dagger k} b^k \rangle, \\ v_k &= \langle v^\dagger v b^{\dagger k} b^k \rangle, & k = 0, 1, 2, \dots \quad \text{and} \\ t_k &= -ig \langle v^\dagger c b^{\dagger k} b^{k-1} \rangle, & k = 1, 2, 3, \dots \end{aligned} \quad (3)$$

Obviously, one has $n_c = \langle c^\dagger c \rangle = c_0$ and $n_v = \langle v^\dagger v \rangle = v_0$,

and $c_0 + v_0 = 1$. Of particular interest are the purely photonic expectation values

$$p_k = c_k + v_k = \langle b^{\dagger k} b^k \rangle, \quad (4)$$

and especially the low-index ones $p_1 = n$, and p_2 related to the autocorrelation function by $p_2 = g^{(2)}(0) p_1^2$.

With the imaginary unit as a prefactor, the multi-photon-assisted polarization t_k is real. From Eq. (1) the equations of motion for these expectation values can be derived, together with the steady-state conditions

$$\frac{\partial}{\partial t} c_k = -(k\kappa + \gamma)c_k + P v_k - 2t_{k+1} = 0, \quad (5a)$$

$$\frac{\partial}{\partial t} v_k = \gamma c_k - (k\kappa + P)v_k + 2t_{k+1} + 2kt_k = 0, \quad (5b)$$

$$\frac{\partial}{\partial t} t_k = g^2 c_k + g^2 k c_{k-1} - g^2 v_k - \frac{(2k-1)\kappa + P + \gamma}{2} t_k = 0. \quad (5c)$$

By adding Eqs. (5a) and (5b) one obtains the steady-state balance relation between the cavity losses and its feeding through the photon-assisted polarization

$$\kappa p_k = 2 t_k, \quad k \geq 1. \quad (6)$$

Using this condition, Eq. (5a) for $k = 0$ can be written as $P n_v = \gamma n_c + \kappa n$, which leads to

$$n_c = \frac{P - \kappa n}{P + \gamma}. \quad (7)$$

This shows that the knowledge of the photon output n is sufficient for the determination of the level occupancies.

More generally, from Eqs. (5a) and (5b) for arbitrary k , the unknowns c_k and v_k can be eliminated in favour of t_k , and using again the balance condition Eq. (6) one obtains closed equations for the photonic expectation values p_k , in the form of a three-term recursion relation, first obtained by Agarwal and Dutta Gupta [16]

$$A_k p_{k+1} + B_k p_k - C_k p_{k-1} = 0, \quad k \geq 1, \quad (8)$$

with

$$\begin{aligned} A_k &= \frac{2\kappa}{k\kappa + P + \gamma}, \\ B_k &= \frac{k\kappa - P + \gamma}{k\kappa + P + \gamma} + \frac{k\kappa}{(k-1)\kappa + P + \gamma} \\ &\quad + \kappa \frac{(2k-1)\kappa + P + \gamma}{4g^2}, \\ C_k &= \frac{kP}{(k-1)\kappa + P + \gamma}. \end{aligned} \quad (9)$$

This is an important feature of the model, allowing direct access to the photon statistics after the elimination of the information regarding the atom subsystem.

Solving Eq. (8) recursively requires knowledge of two initial conditions, and the difficult point is that only one

is available, namely $p_0 = 1$. On the other hand, it is known [29] that the three-term recursion problem can be addressed using continued fraction techniques [16, 30]. To this end Eq. (8) is rewritten as

$$r_k + \lambda_k - \frac{\mu_k}{r_{k-1}} = 0, \quad k \geq 1, \quad (10)$$

with $r_k = p_{k+1}/p_k$, $\lambda_k = B_k/A_k$ and $\mu_k = C_k/A_k$.

As before, the initial condition $r_0 = n$ is not known. On the other hand it turns out that the large k limit is easy to obtain. Indeed, according to their definitions all r_k should be positive and from Eq. (10) the positivity requirement for both r_{k-1} and r_k leads to

$$0 \leq r_{k-1} \leq \frac{\mu_k}{\lambda_k} = \frac{C_k}{B_k}. \quad (11)$$

An examination of Eq. (9) shows that the upper bound goes to zero as $k \rightarrow \infty$, and therefore $\lim_{k \rightarrow \infty} r_k = 0$.

With this 'initial' condition at $k = \infty$, Eq. (10) is solved by a backward propagation of the recursion relation, $r_{k-1} = \mu_k/(\lambda_k + r_k)$. This leads to a continued fraction expansion of the solution, in terms of the parameters μ_k and λ_k

$$\begin{aligned} r_0 &= \frac{\mu_1}{\lambda_1 + r_1} = \frac{\mu_1}{\lambda_1 + \frac{\mu_2}{\lambda_2 + r_2}} = \dots \\ &= \frac{\mu_1}{\lambda_1 + \frac{\mu_2}{\lambda_2 + \frac{\mu_3}{\lambda_3 + \dots}}} \end{aligned} \quad (12)$$

The continued fraction is convergent [31] and the result is numerically in perfect agreement with the long time limit of the Liouville-von Neumann solution (see Fig.1a).

Approximate algebraic expressions $r_0^{(k)}$ for the solution r_0 are obtained by setting in Eq. (12) $r_k = 0$. Fig.2a shows a comparison of the full solution $n = r_0$ with its second approximant $r_0^{(2)}$ which is a rational function of P whose expression follows straightforwardly from Eq. (9) and $r_0^{(2)} = C_1 B_2 / (B_1 B_2 + A_1 C_2)$. The corresponding approximants for n_c obtained by Eq. (7) are shown in linear (Fig.2b) and semi-log (Fig.2c) plots respectively. The agreement is remarkable on the whole range of pump values in the case when the transition is absent (Fig.2d), and when the transition does take place, the agreement remains good for the pump values away from the linear middle interval.

A closer inspection of the numerics shows that inside this interval the continued fraction becomes much more slowly convergent. The reason for this is the fact that the first values of B_k , and with them those of λ_k , are negative, resulting in negative values for the low-ranking approximants of the positive quantity r_0 . Indeed, from Eq. (9) it is clear that B_1 may take negative values due to its first term. For this to happen P should not be too

small. On the other hand for large values of P the last term becomes dominant, making B_1 positive again. In other words, B_1 can be negative but only for an appropriate choice of the parameters κ, γ and g , and only for intermediate pump values. This makes the analysis of the linear middle interval look rather complicated.

Surprisingly, the numerical evidence shows that the situation is quite simple. One remarks that the full solution obeys to a good accuracy the approximation provided by the ansatz

$$r_0 = -\lambda_0 = -\frac{B_0}{A_0} = \frac{P - \gamma}{2\kappa} - \frac{(P + \gamma)(P + \gamma - \kappa)}{8g^2}, \quad (13)$$

which leads, using Eq. (7), to a linear expression for the upper-state population

$$n_c = \frac{1}{2} + \frac{\kappa(P + \gamma - \kappa)}{8g^2}. \quad (14)$$

Moreover, in terms of the recursion relation, this ansatz is equivalent to the assumption that Eq. (8) would hold *even for* $k = 0$ with the last term $C_0 p_{-1}$ supposedly vanishing. Such an interpretation was put forward by del Valle and Laussy [15], based on the argument that $C_0 = 0$. But this definitely requires a more careful analysis, because on the one hand C_0 comes multiplied by the ill-defined p_{-1} and on the other hand the ansatz is obviously not working for all parameter values.

Nevertheless, it is clear (see Fig.2) that the values given by Eqs. (13) and (14) describe successfully the linear middle regime. Moreover, since this regime is supposedly connected with lasing, it is important to clarify the deeper meaning of the ansatz. In other words one has to answer two basic questions: (i) if indeed Eq. (8) can be extended for $k = 0$, which is the right interpretation of its last term in this case and (ii) why is this term vanishing for parameters corresponding to the linear middle regime.

The first question is answered in the next Section and the second in Section VI.

Before closing this Section two remarks are in order. First, it is easy to see that if B_1 is positive, all B_k are positive too, and then the continued fraction coefficients are positive. In this case [29] the odd approximants $r_0^{(2k+1)}$ form a sequence of decreasing upper bounds for the true solution r_0 , while the even approximants $r_0^{(2k)}$ give rise to an increasing sequence of lower bounds. This provides an efficient control of the truncation error, when the system is away from the linear middle regime.

Second, we show that the continued fraction solution provides exact analytic expressions for the autocorrelation function for both the low- and the large-pump limit. One has

$$g^{(2)}(0) = \frac{p_2}{p_1^2} = \frac{r_1}{r_0}, \quad (15)$$

with r_0 as in Eq. (12) and similarly r_1 given by

$$r_1 = \frac{\mu_2}{\lambda_2 + \frac{\mu_3}{\lambda_3 + \dots}}. \quad (16)$$

As $P \rightarrow 0$ all μ_k vanish linearly while λ_k stay finite. Therefore, in this limit one has

$$g^{(2)}(0) = \frac{\mu_2 \lambda_1}{\mu_1 \lambda_2} = \frac{C_2 B_1}{C_1 B_2}, \quad (17)$$

and as a result one obtains

$$g^{(2)}(0)|_{P=0} = 2 \frac{\kappa + \gamma}{3\kappa + \gamma} \frac{4g^2 + \kappa\gamma}{4g^2 + \kappa(\kappa + \gamma)}. \quad (18)$$

In the large P asymptotics $\mu_k \sim Pk/2\kappa$ grow linearly with P and therefore are dominated by the quadratically increasing $\lambda_k \sim P^2/8g^2$. In this case too the asymptotic behaviour is given by Eq. (17) with the outcome $\lim_{P \rightarrow \infty} g^{(2)}(0) = 2$. These facts are borne out by the numerical results.

IV. GLAUBER-SUDARSHAN REPRESENTATION

An alternative approach to the photon statistics is based on the expansion of operators in the photonic Hilbert space as integrals over coherent-state projectors [24]. This is known as the Glauber-Sudarshan (GS) or phase-space representation.

In our case the Hilbert space is enlarged by the QD degrees of freedom and the density operator can be represented as a 2×2 matrix made of blocks acting on the photonic degrees of freedom. For each of these blocks the GS expansion can be used

$$\rho = \begin{pmatrix} \rho^{cc} & \rho^{cv} \\ \rho^{cv} & \rho^{vv} \end{pmatrix} = \int \frac{d\alpha}{\pi} |\alpha\rangle \begin{pmatrix} R_c & \alpha Q \\ \alpha^* Q^* & R_v \end{pmatrix} \langle \alpha|. \quad (19)$$

The GS functions R_c and R_v representing ρ^{cc} and respectively ρ^{vv} , are in principle functions of the complex argument α . Nevertheless, as discussed in Sec. (III) the elements of the density matrix which are diagonal in the QD-level index are diagonal in the photon number too, and this entails that the corresponding GS functions depend only on the radial parameter $s = |\alpha|^2$ and are angle-independent $R_c = R_c(s)$, $R_v = R_v(s)$. Similarly, the off-diagonal block ρ^{cv} has non-zero entries only one step above the diagonal. This corresponds to a GS function with the structure $\alpha Q(s)$. Also, in accordance with Eq. (3) it is convenient to introduce $T(s) = -igQ(s)$, which will turn out to be real.

The advantage of using the GS representation becomes obvious after expressing the expectation values of Eq. (3) in this formalism. Indeed, the procedure is convenient

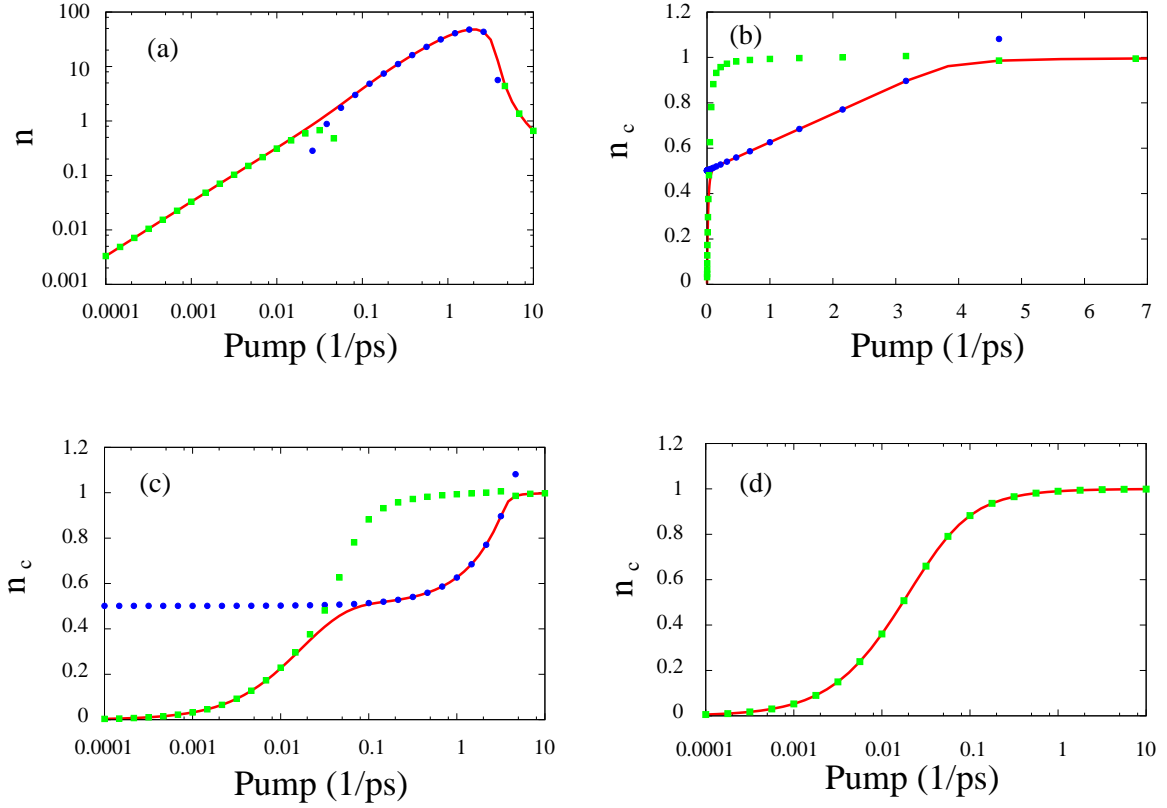


FIG. 2. Comparison between the full solution (red line), the second approximant to the continued fraction (green squares) and the linear middle ansatz, Eqs. (13-14) (blue circles). The parameters are the same as in Fig.1: Set A for panels (a), (b), (c), and Set B for panel (d).

for calculating averages of normal-ordered photonic operators [20, 24]. One obtains

$$\begin{aligned} c_k &= \int_0^\infty R_c(s) s^k ds, \quad v_k = \int_0^\infty R_v(s) s^k ds, \\ t_k &= \int_0^\infty T(s) s^k ds. \end{aligned} \quad (20)$$

This shows that the quantities of interest are moments of the GS functions, and it is clear that this allows to extend their definition. For instance, now it is possible to define t_0 according to Eq. (20), whereas in Eq. (3) it made no sense.

The purely photonic expectation values are obtained, as in Eq. (4), by tracing over the QD-indices

$$p_k = c_k + v_k = \int_0^\infty R(s) s^k ds, \quad (21)$$

in which $R(s) = R_c(s) + R_v(s)$. For further reference let us remind here that the photon number probability distribution $\rho_{n,n} = \rho_{n,n}^{cc} + \rho_{n,n}^{vv}$ is given in terms of $R(s)$ by

$$\rho_{n,n} = \int_0^\infty R(s) \frac{s^n}{n!} e^{-s} ds. \quad (22)$$

The Poissonian statistics is generated by a δ -like GS function, $R(s) = \delta(s - s_0)$, which corresponds in terms of the photonic expectation values p_k to an exponential behaviour $p_k = s_0^k$.

The translation of the Liouville-von Neumann Eq. (1) into the phase-space language leads to Fokker-Planck equations for the GS functions [24], obtained by the correspondence $b \rightarrow \alpha$ and $b^\dagger \rightarrow \alpha^* - \partial/\partial\alpha$ when the photonic operators act on ρ from the left and $b \rightarrow \alpha - \partial/\partial\alpha^*$ and $b^\dagger \rightarrow \alpha^*$ when they act from the right. Taking into account that the GS functions appearing here depend only on the radial argument, one obtains the counterpart of Eqs. (5) in the form

$$\begin{aligned} \frac{\partial}{\partial t} R_c &= -2sT + \kappa(R_c + sR'_c) \\ &\quad - \gamma R_c + PR_v = 0, \end{aligned} \quad (23a)$$

$$\begin{aligned} \frac{\partial}{\partial t} R_v &= -2sT' + 2(s-1)T + \kappa(R_v + sR'_v) \\ &\quad + \gamma R_c - PR_v = 0, \end{aligned} \quad (23b)$$

$$\begin{aligned} \frac{\partial}{\partial t} T &= g^2(R_c - R_v) - g^2R'_c + \kappa\left(\frac{3}{2}T + sT'\right) \\ &\quad - \frac{\gamma + P}{2}T = 0, \end{aligned} \quad (23c)$$

where the prime stands for the derivative with respect to

s . By adding Eqs. (23a) and (23b) one obtains successively

$$-2(sT)' + \kappa(sR)' = 0, \quad s\kappa R = 2sT, \quad (24)$$

wherefrom the analogue of the balance condition Eq. (6) follows

$$\kappa R = 2T. \quad (25)$$

The link to the equations of motion Eqs. (5) is obtained by taking the k -th moment of both sides of the Fokker-Planck Eqs. (23) and using integration by parts for the terms containing derivatives. The advantage of this approach is that not only the equations of motions are recovered, but now the $k = 0$ version of the last one (Eq. (5c)) can be obtained, and reads

$$g^2 c_0 - g^2 v_0 + g^2 R_c(0) - \frac{-\kappa + \gamma + P}{2} t_0 = 0, \quad (26)$$

which amounts to extending the interpretation of kc_{k-1} for $k = 0$ by the value of the GS function $R_c(s)$ at the origin. The process of eliminating c_k and v_k in favour of t_k can be repeated, but now including the $k = 0$ case. The balance relation Eq. (6) holds for $k = 0$ too, owing to its GS version Eq. (25). As a result one recovers the three-term recursion formula Eq. (8) extended for $k = 0$ as

$$A_0 p_1 + B_0 p_0 - \frac{P}{-\kappa + \gamma + P} R(0) = 0. \quad (27)$$

This shows that $R(0)$ is the correct interpretation of kp_{k-1} for $k = 0$ [32]. In deriving the last equation we made use of Eq. (23a) which, for $k = 0$ and $s = 0$, provides the relationship between $R_c(0)$ and $R_v(0) = R(0) - R_c(0)$, allowing to express $R_c(0)$ in terms of $R(0)$.

Having established the generalization of the recursion relation to $k = 0$, it becomes clear that the ansatz contained in Eq. (13), and which leads to the linear middle behaviour Eq. (14), is equivalent to the vanishing of $R(0)$. We will therefore be concerned in what follows with the circumstances in which this vanishing may take place.

One way to address this problem is to eliminate $R_c(s)$ and $R_v(s)$ from Eqs. (23) in order to set up a differential equation for $T(s)$ which, through Eq. (25), provides a closed equation for the purely photonic GS function $R(s)$. This essentially parallels the procedure of obtaining the recursion relation, except that we perform it at the level of the GS functions and not of their moments. One obtains in this way a second-order differential equation of the form

$$(as^2 + bs^3)R''(s) + (c + ds + es^2)R(s) + (u + vs + ws^2)R(s) = 0. \quad (28)$$

Details about the derivation of this equation and the expression of the coefficients in terms of the model parameters κ , γ and the pump rate P are given in the Appendix A.

The structure of this equation corresponds to $s = 0$ being an irregular singular point [33]. Only one of the fundamental solutions behaves around $s = 0$ as a power, $R(s) \sim s^\sigma$ and the analysis leads to the result $\sigma = 0$. The other fundamental solution is highly singular $R(s) \sim \exp(\lambda/s)$, $\lambda > 0$ [33]. These conclusions are parameter independent and show that one cannot find a solution which vanishes at the origin.

This seems to contradict the previously mentioned numerical findings. The paradox comes from the requirement $R(0) = 0$, which is too strong. In fact the numerical evidence hints only at $R(0)$ being negligibly small. We show below that, indeed, the strong condition $R(0) = 0$ holds only as a certain limiting case, which corresponds to a sharp laser transition. Away from this limit the transition gets smoother and the condition is expected to hold only as an approximation. This is a familiar situation, also encountered in connection with the many-emitter lasers [27]. The scaling limit procedure found in that context is based on an analysis performed at the level of the laser rate equation. In the present case too, we first take guidance from the rate equation approximation to find the appropriate limiting regime, which we later analyze at the level of the exact solution.

V. THE RATE EQUATION APPROXIMATION

The infinite hierarchy of the equations of motion Eq. (5) can be interrupted by assuming that higher-order expectation values can be approximately factorized into products of lower-order ones. The factorization leads to the rate-equation approximation to the full Liouville-von Neumann problem [27]. In our case, using

$$\begin{aligned} c_1 &= \langle c^\dagger c b^\dagger b \rangle \approx \langle c^\dagger c \rangle \langle b^\dagger b \rangle = c_0 p_1, \\ v_1 &= \langle v^\dagger v b^\dagger b \rangle \approx \langle v^\dagger v \rangle \langle b^\dagger b \rangle = v_0 p_1, \end{aligned} \quad (29)$$

and limiting ourselves to the steady state, we are left with the following closed set of equations for the quantities of interest $n_c = c_0$, $n_v = v_0$ and $n = p_1$

$$\begin{aligned} n_c + n_v &= 1, \\ -\gamma n_c + P n_v &= \kappa n, \\ \Gamma n_c (n + 1) - \Gamma n_v n &= \kappa n. \end{aligned} \quad (30)$$

Here $\Gamma = 4g^2/(\kappa + \gamma + P)$ is the rate of spontaneous emission into the laser mode, and the physical interpretation of different terms is obvious. Let us mention only the fact that the decrease of Γ with P is a reflex of the dephasing effect of the pump, which leads eventually to the quenching of photon output.

Solving the first two equations for the level occupancies in terms of the photon number leads to

$$n_c = \frac{P - \kappa n}{\gamma + P}, \quad n_v = \frac{\gamma + \kappa n}{\gamma + P}, \quad (31)$$

which, introduced in the last one, gives rise to a quadratic equation for n

$$2(\kappa n)^2 + \left[\kappa + \gamma - P + \frac{\kappa(\gamma + P)(\gamma + \kappa + P)}{4g^2} \right] (\kappa n) - \kappa P = 0, \quad (32)$$

The two solutions have opposite signs and, of course, only the positive one is physical.

For the sake of the following discussion the above equation was rewritten in terms of the rescaled photon number $\tilde{n} = \kappa n$. In this form, in the limit $\kappa \rightarrow 0$ the free term disappears and one of the roots becomes $\tilde{n} = 0$. This vanishing solution holds in the parameter domain in which the other root is negative but, when positive, it is this second root that is retained. Therefore an abrupt change of behaviour takes place when the non-zero solution changes sign. As a function of the pump rate the second root is a quadratic polynomial, initially increasing but then decreasing, as predicted by the quenching effect. This behaviour is preserved even in the $\kappa \rightarrow 0$ limit, provided one takes simultaneously $g \rightarrow 0$ in such a way that $\tilde{g}^2 = g^2/\kappa$ stays finite. In this limit the solution reads

$$\tilde{n} = \frac{P - \gamma}{2} - \frac{(\gamma + P)^2}{8\tilde{g}^2} = \nu(P) \quad (33)$$

when $\nu(P) \geq 0$ and $\tilde{n} = 0$ otherwise. For the upper-state population this leads to

$$n_c = \begin{cases} \frac{P}{\gamma + P} & \text{if } \nu(P) < 0 \\ \frac{1}{2} + \frac{\gamma + P}{8\tilde{g}^2} & \text{if } \nu(P) \geq 0 \end{cases} \quad (34)$$

These solutions are shown in Fig.3. It is clear that the concave dependence of n_c on P is interrupted by a linear middle behaviour, in the interval defined by the roots of $\nu(P)$.

The situation described by the rate equation approximation suggests that, in the above-defined scaling limit, the interval of positive values for $\nu(P)$ coincides with the interval in which $R(0) = 0$, and in which the linear middle ansatz Eq. (13) holds. This is indeed the case, as will be proven in Sec. VI.

Several comments are in order. First, the appearance of the transition is conditioned by existence of real, positive roots of the equation $\nu(P) = 0$. It is easy to see that real roots exist if $\tilde{g}^2 \geq 2\gamma$, or $g^2 \geq 2\kappa\gamma$, and then they are both positive. The two types of behaviour discussed in Fig.1 are distinguished by whether this condition is fulfilled or not.

Second, the scaling procedure defined here $\kappa \rightarrow 0$, $g \rightarrow 0$, g^2/κ finite, corresponds entirely to the limiting regime ($\kappa \rightarrow 0$, $\beta \rightarrow 0$, β/κ finite) considered by Rice and Carmichael [27], because the β -factor is proportional to g^2 . Moreover, the Jaynes-Cummings coupling constant appears as an explicit parameter in all laser models and therefore the scaling limit formulated in terms of g is not limited to the two-level atom model (see Appendix B).

Finally, since \tilde{n} is the rate of photon emission out of the cavity, its nonvanishing value as κ goes to zero expresses the appearance of a finite laser output even as we get arbitrarily close to a perfect cavity, by a corresponding increase of the photon generation inside. This is physically interpreted as “an explosion of stimulated emission” [27], typical for the lasing regime.

The rate equation approach provides only partial informations about the photon system. The full photon statistics is contained in the GS function $R(s)$, whose scaling behaviour is addressed in the next Section.

VI. THE SCALING LIMIT OF $R(s)$

As $n \rightarrow \infty$, the moments of $R(s)$ cease to exist, showing that the GS function tends to move its weight on larger and larger s values. In this situation, meaningful quantities are the rescaled ones, like

$$\tilde{n} = \tilde{p}_1 = \int_0^\infty \kappa R(s) s ds. \quad (35)$$

In the rescaled variable $t = \kappa s$, one has

$$\tilde{p}_1 = \int_0^\infty \tilde{R}(t) t dt, \quad (36)$$

where the rescaled GS function $\tilde{R}(t)$ is defined by

$$\tilde{R}(t) = \frac{1}{\kappa} R\left(\frac{t}{\kappa}\right). \quad (37)$$

Similarly, all the higher moments p_k are rescaled as $\tilde{p}_k = \kappa^k p_k$ and become corresponding moments of $\tilde{R}(t)$.

$$\tilde{p}_k = \int_0^\infty \tilde{R}(t) t^k dt. \quad (38)$$

The task now is to establish the equation obeyed by $\tilde{R}(t)$ in the scaling limit and to study the behaviour of its solution around the origin. Intuitively, according to Eq. (37), the graph of $\tilde{R}(t)$ is obtained from that of $R(s)$ by compressing the latter by a factor of $1/\kappa$ along the abscissa and expanding it by the same factor along the ordinate. This would bring the rescaled function to $\delta(t)$, in the limit $\kappa \rightarrow 0$, were it not for the opposite tendency of $R(s)$ to move away from the origin, as discussed above. The net result of these competing trends is analyzed below.

To this end, one translates Eq. (28) obeyed by $R(s)$ into an equation for $\tilde{R}(t)$, retaining in the resulting coefficients of the latter only the dominant terms in the scaling parameter κ . One obtains [34]

$$\begin{aligned} \frac{t^2}{4\tilde{g}^2} \kappa^2 \tilde{R}'' - \left[\left(3 \frac{\gamma + P}{8\tilde{g}^2} + 1 \right) t - \frac{1}{2} P \right] \kappa \tilde{R}' \\ + [t - \nu(P)] \tilde{R} = 0. \end{aligned} \quad (39)$$

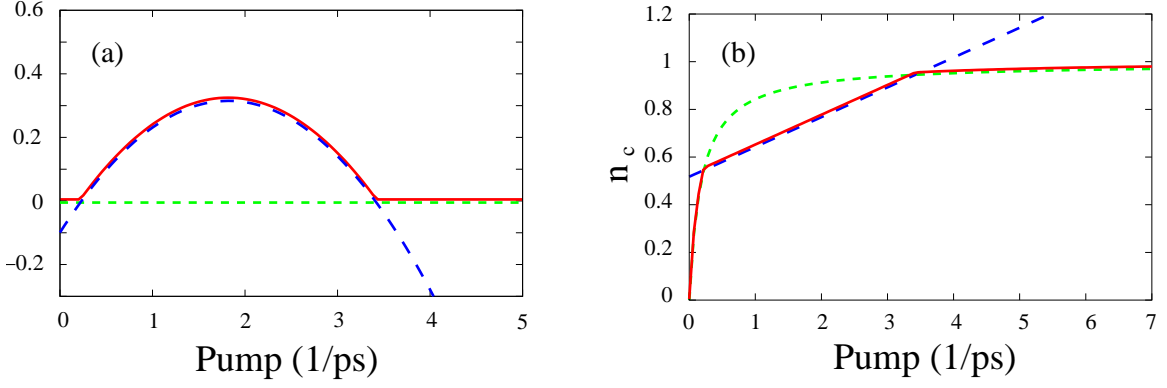


FIG. 3. (a) \tilde{n} (full red line) is equal to $\nu(P)$ (blue dashed line) when the latter is positive and zero (green dashed line) otherwise. (b) the values of the upper level population n_c for these cases, Eq. (34), showing a linear middle behaviour.

The appearance of the small parameter κ with the derivatives suggests a WKB approach to the $\kappa \rightarrow 0$ asymptotics of the solution. In other words one searches the solution, up to a normalization factor, in the form

$$\tilde{R}(t) = e^{-\frac{1}{\kappa}\varphi(t)}, \quad (40)$$

in which $\varphi(t)$ is taken in the leading, zeroth order in κ . It is clear that when κ gets smaller, the value of $\tilde{R}(t)$ around the minimum of $\varphi(t)$ is greatly enhanced, at the expense of the values at other points which, in the view of normalization, become negligible. In the limit one obtains a δ -function concentrated at the minimum of $\varphi(t)$.

The equation obeyed by $\varphi(t)$ in the leading order has the form of a quadratic equation for its derivative

$$\frac{t^2}{4\tilde{g}^2}(\varphi')^2 + \left[\left(3\frac{\gamma+P}{8\tilde{g}^2} + 1 \right) t - \frac{1}{2}P \right] \varphi' + [t - \nu(P)] = 0. \quad (41)$$

Around $t = 0$ one of the roots behaves like $\varphi' \sim 2\tilde{g}^2 P/t^2$, i.e. $\varphi \sim -2\tilde{g}^2 P/t$ which, by Eq. (40), leads to the singular solution mentioned in Section IV. The regular solution comes from the other root, for which $\varphi'(0) = -2\nu(P)/P$.

Two cases arise, depending on the sign of $\nu(P)$: (i) As long as $\nu(P)$ is negative one has $\varphi'(0) > 0$ and $t = 0$ is a minimum for $\varphi(t)$. Then, according to the above discussion, $\tilde{R}(t)$ tends to $\delta(t)$ in the scaling limit. (ii) When $\nu(P)$ becomes positive, φ' starts at $t = 0$ with negative values and crosses the abscissa at $t = \nu$. Then $\varphi(t)$ has a local maximum at the origin and therefore the values of $\tilde{R}(0)$ become vanishingly small in the limit $\kappa \rightarrow 0$. This is precisely the requirement for the linear middle behaviour to hold. The values of $\tilde{R}(t)$ are now concentrated at the point of minimum $t = \nu(P)$. In other words $\tilde{R}(t) \rightarrow \delta(t - \nu)$.

In the first case, according to Eq. (38), all rescaled expectation values are vanishing in the scaling limit, except $\tilde{p}_0 = 1$, while in the second regime they become $\tilde{p}_k = \nu^k$. This exponential behaviour of the normal-ordered averages $\langle b^{\dagger k} b^k \rangle$ is equivalent to a Poissonian photon number

statistics. The change between the two regimes occurs when $\nu(P)$ changes sign, from negative to positive and back, in agreement with the result suggested by the rate equation approximation.

In terms of the phase-space complex argument α , the rescaled GS function $\tilde{R}(t)$ changes during the transition from being concentrated around the origin into a infinitely thin ring-like distribution having a radius of $\sqrt{\nu(P)}$. For the unscaled GS function $R(s)$, this corresponds to a ring distribution whose radius grows in the scaling limit faster than its thickness [35]. This is how its value at the origin becomes negligibly small, in accordance with the discussion in Section III regarding why and when is the ansatz of Eq. (13) valid.

VII. CONCLUSIONS AND DISCUSSION

The numerical solution of the problem of a two-level system in interaction with a cavity mode shows a rather sudden change in the behaviour of the steady-state data as a function of pumping. Using an analytic approach, we reproduce the numerical results and discuss this regime change.

One of the main tools used was the Glauber-Sudarshan representation, which was shown to provide the natural extension, Eq. (27), to the recursion relation Eq. (8). In this perspective it becomes clear that the transition observed is related to the vanishing of the GS quasi-distribution function $R(s)$ at the origin.

The laser transition is often represented as the tendency of the GS function to move away from the origin, and this is exactly what takes place in this case too. More precisely, it was shown that in the limit $\kappa \rightarrow 0$, $g \rightarrow 0$ so that $g^2/\kappa = \text{const}$, the GS function becomes in the lasing regime a δ -like distribution concentrated on a ring with a pump-dependent radius. In terms of photon number statistics this means that the distribution becomes Poissonian.

It is instructive to see the action of the scaling limit

directly on the recursion relation Eq. (8). Rewritten for the rescaled quantities $\tilde{p}_k = \kappa^k p_k$ its coefficients are changed respectively into $\tilde{A}_k = A_k/\kappa$, $\tilde{B}_k = B_k$ and $\tilde{C}_k = \kappa C_k$. In the scaling limit these coefficients become k -independent

$$\begin{aligned}\tilde{A}_k &\rightarrow \frac{2}{P+\gamma}, \\ \tilde{B}_k &\rightarrow \frac{-P+\gamma}{P+\gamma} + \frac{P+\gamma}{8\tilde{g}^2}, \\ \tilde{C}_k &\rightarrow 0,\end{aligned}\tag{42}$$

with the ratio \tilde{A}_k/\tilde{B}_k converging to $\nu(P)$. Then the recursion relation becomes simply $\tilde{p}_{k+1} = \nu \tilde{p}_k$ for $k \geq 1$. For the values of the pump for which ν is negative only the trivial solution $\tilde{p}_k = 0$ for all $k \geq 1$ is allowed, to avoid the appearance of negative expectation values. Conversely, when $\nu(P)$ is positive a nontrivial solution becomes possible. Moreover, according to the above discussion, in this case $\tilde{R}(0) = 0$ and then the additional relation $\tilde{p}_1 = \nu \tilde{p}_0 = \nu$, deduced from Eq. (27) also holds. Then the solution is $\tilde{p}_k = \nu^k$, in agreement with the statement that one has $\tilde{R}(t) \rightarrow \delta(t - \nu)$ in the scaling limit and the photon statistics becomes purely Poissonian.

Examining the convergence in Eq. (42) it is clear that as κ becomes smaller and smaller it is the low- k coefficients that get close to their limit first. This is because the limit requires the product $k\kappa$ to be small. Therefore the statistics becomes accurately Poissonian for the low-index p_k first. As κ further approaches zero the list of nearly-Poissonian expectation values grows longer and longer and only in the limit it extends to the whole sequence. In view of these facts $g^{(2)}(0)$ gets close to unity quite early in the process, before the statistics becoming entirely Poissonian. As a symptom of a truly coherent light, the condition $g^{(2)}(0) = 1$ may be, in this sense, somewhat premature.

The scaling limit corresponds to a weak coupling of the cavity both to its photon source and to its drain. Both coupling parameters involved are present in all laser models, so that the limiting procedure can be tested, in principle, on any of them, and we have shown its validity of the Scully-Lamb model too. As a procedure which is successful in two unrelated cases, the scaling limit has a good chance of having a wider applicability.

Appendix A: Differential equation for the Glauber-Sudarshan function $R(s)$

The second order differential equation obeyed by $R(s)$ is deduced from Eqs. (23) as follows. We note first that R_v and T can be eliminated in favour of R and R_c , the first by $R_v = R - R_c$ and the second using the balance condition Eq. (25). As a second step one solves for R_c and R'_c in favour of R and then identifies the derivative of the first with the second. This way Eq. (28) is obtained,

with the following coefficients

$$\begin{aligned}a &= \kappa \frac{\gamma - \kappa + P}{g^2}, \\ b &= -2 \frac{\kappa^2}{g^2}, \\ c &= \frac{2P(\gamma - \kappa + P)}{\kappa^2}, \\ d &= - \left[4 \frac{P}{\kappa} + \frac{\gamma - \kappa + P}{\kappa} \left(4 + 3\kappa \frac{\gamma - 3\kappa + P}{2g^2} \right) \right], \\ e &= 8 + \kappa \frac{3\gamma - 7\kappa + 3P}{g^2}, \\ u &= - \frac{4P(\gamma - 2\kappa + P)}{\kappa^2} \\ &\quad + \frac{\gamma - \kappa + P}{\kappa} \left[2 \frac{\gamma - \kappa + P}{\kappa} + \frac{(\gamma - 2\kappa + P)(\gamma - 3\kappa + P)}{2g^2} - 4 \right], \\ v &= 8 \frac{P}{\kappa} - (\gamma - \kappa + P) \frac{\gamma - 3\kappa + P}{g^2}, \\ w &= -8.\end{aligned}\tag{A1}$$

Appendix B: The random injection model

In this Appendix we apply the scaling limit to the random injection model, in order to show that in this case too a sharp transition to Poissonian distribution takes place, in the same parameter region as given in the literature.

In the Scully-Lamb laser model [17, 19, 21] the cavity mode is fed by random injection of inverted two-level atoms, which exchange energy with the mode through Jaynes-Cummings interaction. The time each atom spends inside the cavity is a random variable too. The losses are simulated by a similar random injection of atoms, this time in their lower state. One obtains for the time evolution obeyed by the photonic density matrix

$$\begin{aligned}\frac{\partial}{\partial t} \rho_{n,n} &= \frac{A n}{1 + \frac{B}{A} n} \rho_{n-1,n-1} - \frac{A(n+1)}{1 + \frac{B}{A}(n+1)} \rho_{n,n} \\ &\quad - C n \rho_{n,n} + C(n+1) \rho_{n+1,n+1}.\end{aligned}\tag{B1}$$

We keep here the notation commonly found in the literature [17, 19, 20] for the resulting parameters. For $B = 0$ one would obtain the usual competition between the loss factor C and the gain (or amplification) factor A . This is known to lead to an unstable behaviour for $A > C$. The role of the saturation parameter $B > 0$ is to limit the photon output and to stabilize the solution. A common procedure in the analysis of the problem is to expand the denominators of Eq. (B1) up to the first order in B , but we will not resort to this approximation.

In terms of the Jaynes-Cummings coupling constant one has A and C proportional to g^2 and B to g^4 . Since the cavity loss rate C is the same as the parameter κ used throughout the paper, the limit $\kappa \rightarrow 0$ naturally implies that $g^2 \rightarrow 0$ too, with their ratio remaining constant.

It is easily seen that the steady-state solution of Eq. (B1) obeys

$$\rho_{n+1,n+1} = \frac{\mu}{\beta+1+n} \rho_{n,n}, \quad (\text{B2})$$

where the notation $\mu = A^2/BC$ and $\beta = A/B$ has been used. This leads to

$$\rho_{n,n} = \frac{\mu^n}{(\beta+1)(\beta+2)\dots(\beta+n)} \rho_{0,0}. \quad (\text{B3})$$

As it was early recognized [17], the generating function of $\rho_{n,n}$ is given, up to normalization, by Kummer's confluent hypergeometric function

$$\begin{aligned} \rho(x) &= \sum_n \rho_{n,n} x^n = \rho_{0,0} \sum_n \frac{(\mu x)^n}{(\beta+1)(\beta+2)\dots(\beta+n)} \\ &= \rho_{0,0} {}_1F_1(1, \beta+1, \mu x). \end{aligned} \quad (\text{B4})$$

The value of $\rho_{0,0}$ is determined by the normalization condition $\rho(x) = 1$. The photon number distribution $\rho_{n,n}$ is connected to the GS function $R(s)$ by Eq. (22) which implies, with $y = 1 - x$ and $z = y/\kappa$

$$\begin{aligned} \rho(x) &= \int_0^\infty R(s) e^{-ys} ds = \sum_{k=0}^\infty \frac{(-1)^k}{k!} y^k p_k \\ &= \sum_{k=0}^\infty \frac{(-1)^k}{k!} z^k \tilde{p}_k. \end{aligned} \quad (\text{B5})$$

This shows that $\rho(x)$ is also the generating function for the expectation values p_k and of the rescaled ones \tilde{p}_k when seen as a function of y and z , respectively. Therefore $\rho(x)$ is the appropriate tool for analyzing the scaling behaviour of these quantities. According to their definition, both μ and β scale like $1/g^2$ so they go to infinity as κ goes to zero, in such a way that $\tilde{\mu} = \kappa\mu$ and $\tilde{\beta} = \kappa\beta$ remain finite.

At this point one can make use of the integral representation of Kummer's function [36]

$${}_1F_1(1, \beta+1, \mu x) = \beta \int_0^1 e^{\mu x u} (1-u)^{\beta-1} du, \quad (\text{B6})$$

to express the generation function for \tilde{p}_k as

$$\tilde{p}(z) = \rho(1 - \kappa z) = \frac{I(z)}{I(0)}, \quad (\text{B7})$$

with the notation

$$I(z) = \int_0^1 e^{(1-\kappa z)u\tilde{\mu}/\kappa} (1-u)^{\tilde{\beta}/\kappa-1} du, \quad (\text{B8})$$

and division by $I(0)$ is necessary to ensure the normalization condition $\tilde{p}(0) = \rho(1) = 1$. Rearranging $I(z)$ as

$$I(z) = \int_0^1 e^{-zu\tilde{\mu}} e^{[\tilde{\mu}u + \tilde{\beta}\ln(1-u)]/\kappa} (1-u)^{-1} du, \quad (\text{B9})$$

one brings it into a form familiar from the method of steepest descent. The $\kappa \rightarrow 0$ asymptotics of $I(z)$ is controlled by the maximum of $\varphi(u) = \tilde{\mu}u + \tilde{\beta}\ln(1-u)$ in the interval of integration, since the contribution of the integrand around the maximum point becomes overwhelmingly dominant. Depending on the position u_0 of the saddle point, defined by $\varphi'(u_0) = 0$, i.e.

$$u_0 = 1 - \frac{\tilde{\beta}}{\tilde{\mu}} = 1 - \frac{\beta}{\mu} = 1 - \frac{C}{A}, \quad (\text{B10})$$

one has two regimes. (i) When $A < C$, u_0 is negative and the maximum of $\varphi(u)$ on $[0, 1]$ is found at $u = 0$. (ii) When $A > C$ the saddle point enters the integration interval and the maximum of $\varphi(u)$ is reached for $u = u_0$. As a result, and given the normalization condition Eq. (B7), one obtains

$$\tilde{p}(z) = \begin{cases} 1 & \text{if } A \leq C, \\ e^{-z\mu u_0} & \text{if } A > C. \end{cases} \quad (\text{B11})$$

In the first case all $\tilde{p}_k = 0$ with the exception of $\tilde{p}_0 = 1$, while in the second one obtains $\tilde{p}_k = \nu^k$ with

$$\nu = \tilde{\mu} u_0 = \frac{A(A-C)}{BC}. \quad (\text{B12})$$

One finds again the same scenario as in Section VI. The rescaled expectation values \tilde{p}_k obey an exponential k -dependence, with a base that changes from zero to a positive ν at the threshold point. The expression of ν found by the present procedure is the same as given in the literature [17, 19, 20].

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